D PHYS

Physics Lab

ETH zürich

Ultrasound

Abstract

In this experiment we will measure the speed of sound in water. For this purpose, optical methods are employed to measure the wavelength of sound waves with a known frequency from which the velocity can be calculated. The measurements make use of the diffraction method and the schlieren method.

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Contents

1 Sound waves in fluids

1.1 Short overview of the experiment

Ultrasound refers to sound waves with frequencies that lie above the frequencies that can be heard by the human ear (about 20 kHz). Sound waves propagate in matter as pressure waves. In this experiment, pressure waves in a liquid will lead to local mass density modifications that, in turn, will lead to a local change in the refractive index of the liquid.

The sound waves are generated using the piezoelectric effect in a quartz-crystal, which is excited to oscillate by applying an alternating electric field (generated by a high frequency generator). In order to achieve the maximum oscillation amplitude, the quartz will be excited at its resonance frequency. In our case, resonance frequencies of the quartz lie at 1.3 MHz and 2 MHz. To be able to describe the properties of sound waves in liquids, two basic equations from hydrodynamics are needed: the continuity equation, which describes the conservation of mass, and the Euler equation, which is an equation of motion.

1.2 Conservation of mass

The state of a liquid is completely described by a set of three quantities: the pressure $p(x, y, z, t)$, the mass density $\rho(x, y, z, t)$ and the velocity field $\vec{v}(x, y, z, t)$. We will consider p, ρ and $\vec{v}$ belonging to a fixed point in space and not to the position of a certain volume element moving with the liquid. The determination of the vector field $\vec{v}$ requires knowledge of $\mathrm{curl}(\vec{v}) = f(p, \rho)$ and $\mathrm{div}(\vec{v}) = g(p, \rho)$. Here we assume that $\mathrm{curl}(\vec{v}) = 0$ (no curls in the liquid). Two additional equations are then needed to determine ρ and p.

The idea of the following derivation lies in the assumption that mass is neither created nor destroyed in the fluid. If we now consider a volume V_0, the mass of the liquid in this volume is $\int_{V_0} \rho\, dV$. This means that an amount of liquid $\rho\vec{v}d\vec{f}$ is flowing through every surface element $d\vec{f}$ of V_0 per time unit. The entire amount of liquid, which leaves the volume V_0 within a time unit is on one hand

$$\oiint_{\partial V_0} \rho\vec{v}\, d\vec{f}, \tag{1}$$

or on the other hand

$$-\frac{\partial}{\partial t} \iiint_{V_0} \rho\, dV. \tag{2}$$

Equating the two expressions above leads to

$$-\frac{\partial}{\partial t} \iiint_{V_0} \rho\, dV = \oiint_{\partial V_0} \rho\vec{v}\, d\vec{f} \tag{3}$$

Applying Gauss' integral theorem (see Analysis lecture)

$$\oiint_{\partial V_0} \rho\vec{v} d\vec{f} = \iiint_{V_0} \mathrm{div}(\rho\vec{v}) dV,$$

the right hand side of (3) can be changed into a volume integral, and we obtain

$$\iiint_{V_0} \left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\vec{v})\right] dV = 0. \tag{4}$$

This equation is valid for any arbitrary volume V_0, meaning that the integrand needs to vanish. This leads to the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0. \tag{5}$$

1.3 Newton's equation of motion for the fluid

The idea of the next part of the derivation is to find three additional equations by applying Newton's equations of motion to the fluid. For this purpose we are now changing the perspective: we now consider a mass element of the fluid moving with the fluid, in contrast to the previous section. The mass of the fluid element is taken to be constant as the element moves. This means that its volume may change in time.

The force acting on the particles in the fluid element with volume V_0 is given by

$$\vec{F} = -\oint\!\!\oint_{\partial V_0} p\, d\vec{f}, \tag{6}$$

corresponding to the negative integral of the pressure at the surface of the fluid element over the surface of V_0. We convert the surface integral into a volume integral, and obtain

$$\vec{F} = -\oint\!\!\oint_{\partial V_0} p\, d\vec{f} = -\iiint_{V_0} (\nabla p) dV. \tag{7}$$

As we know the force on the particles in a volume element now, we can write Newton's equation of motion for the fluid element

$$\rho \frac{d\vec{v}}{dt} = -\nabla p. \tag{8}$$

The term $d\vec{v}/dt$ describes the change of velocity of a certain package of liquid (*total* or *material* change of velocity), but *not* the change of velocity of a fixed point in space $\partial\vec{v}/\partial t$ (*local* change of velocity). We remember that $\vec{v}$ is a function of x, y, z and t [see eq. (5)]. This is why we have to take the total differential of $\vec{v}$ with respect to all these variables. We obtain

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + \frac{\partial \vec{v}}{\partial x} dx + \frac{\partial \vec{v}}{\partial y} dy + \frac{\partial \vec{v}}{\partial z} dz. \tag{9}$$

Dividing both sides by dt we find

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}. \tag{10}$$

If we insert this result into eq. (8), we obtain the *Euler Equation*

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p. \tag{11}$$

The Euler equation has the inconvenient nature to be non-linear. The linear combination of two or more solutions of this equation is not a solution. This makes this equation difficult to solve.

1.4 Sound waves

A sound wave in a compressible liquid is an oscillating modulation of ρ and p with small amplitudes. In a sound wave the liquid is periodically compressed and expanded. Since the oscillations of a sound wave are assumed to be small, the particle velocity, the density and the pressure oscillations will be small as well. The variables $\vec{v}$, p and ρ can then be written as

$$\vec{v} = \vec{v}_0 + \vec{v}' = \vec{v}', \quad p = p_0 + p', \quad \rho = \rho_0 + \rho', \tag{12}$$

where p_0 and ρ_0 denote the constant equilibrium pressure and the constant equilibrium density of the liquid. The small variations p' and ρ' obey $p' \ll p_0$ and $\rho' \ll \rho_0$.

If one inserts eq. (12) into the continuity equation (5) and neglects terms of second order $\nabla(\rho'\vec{v}')$ (p', ρ' and $\vec{v}'$ being terms of first order), one finds the linearized continuity equation

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0. \tag{13}$$

In analogy, we can derive the linearized Euler equation

$$\frac{\partial \vec{v}'}{\partial t} + \frac{1}{\rho_0} \nabla p' = 0, \tag{14}$$

where the term $(\vec{v} \cdot \nabla)\vec{v}$ has been neglected due to the smallness of $\vec{v}$. The four equations (13) and (14) contain the five unknown functions ρ', p' and $\vec{v}'$. In order to find the fifth equation necessary for finding a solution, we regard the pressure to be a function of density $p(\rho)$. The small change in pressure p' is then found from the small change of density ρ' by the Taylor expansion

$$p' = \left(\frac{\partial p}{\partial \rho}\right)_{\rho=\rho_0} \rho'. \tag{15}$$

The factor $(\partial p/\partial \rho)_{\rho=\rho_0}$ is called compressive modulus. It describes the relationship between pressure and change in density (and the change in volume, as the mass is included) in a medium and it is thus a measure for the elasticity of the medium. If we apply this relation to eq. (13), we obtain

$$\frac{\partial p'}{\partial t} + \rho_0 \left(\frac{\partial p}{\partial \rho_0}\right)_{\rho=\rho_0} \nabla \vec{v} = 0. \tag{16}$$

There are still the unknown functions $\vec{v}$ and p_0. To express these two functions by one single function, it proves to be convenient to introduce the "velocity potential" φ via $\vec{v} = \nabla\varphi$. Thus the linearized Euler equation (14) reduces to

$$p' = -\rho_0 \frac{\partial \varphi}{\partial t}, \tag{17}$$

and we get a relationship between p' and φ. By putting this relation into the linearized continuity equation (16) one finally finds the wave equation

$$\frac{\partial^2 \varphi}{\partial t^2} - c^2 \nabla \varphi = 0. \tag{18}$$

It describes the propagation of sound waves with the velocity

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{\rho=\rho_0}}. \tag{19}$$

The solution of the wave equation are plane waves

$$\varphi\left(\vec{x}\right) = \varphi_0 e^{i\left(\vec{k}\vec{x}-\omega t\right)} \tag{20}$$

where $\omega = c|\vec{k}|$. By inserting this result into eq. (17) and (15) one realizes, that also p' and ρ' are plane waves, which propagate with the velocity c.

Sound waves have the property, that they form standing waves in a basin with a certain shape. These stationary wave patterns are called eigenmodes. They occur resonantly at particular frequencies. In order to tune an eigenmode to a particular frequency, the geometry has to be chosen in a way, that one dimension (e.g. the length) corresponds to an integer multiple of half the wavelength.

If light falls into a basin which is resonantly excited by sound, the standing waves act as an optical grating. Therefore diffraction and scattering phenomena are observable in complete analogy to conventional optical gratings. The corresponding physics will be discussed in the following.

1.5 Diffraction of light waves at an optical grating

If a wave front hits a grating, one observes constructive interference of the scattered waves in discrete directions. In order to understand this phenomenon, we first consider a thin transmission grating which is illuminated by a plane wave from below. Starting from every transparent point in the grating a spherical wave emerges. These spherical waves are interfering with each other. Constructive interference occurs if the path difference Δx of two spherical waves equals $n\lambda$, where n is an integer and λ is the wavelength of the light. If the distance from the grating is very large ($z \gg \lambda$) one can approximate the outgoing beams to be approximately parallel. As seen in Fig. 1, $\Delta x = d \sin \alpha$. Accordingly, for constructive interference the Bragg-condition

$$\sin\left(\alpha\right) = \frac{n\lambda}{d} \tag{21}$$

is valid. In the case of standing waves in a liquid the diffraction grating is not a transmission grating but a phase grating. That means, that no change in amplitude occurs, but only a change in phase (by the periodic modulation of the refractive index). To be able to describe diffraction phenomena in general and quantitatively, one uses the formalism shown below. We first define the grating in the x-z-plane and describe it with the complex-valued transmission coefficient

$$t\left(x\right) = \sum_{n=-\infty}^{\infty} \hat{t}_n e^{2\pi i n x/d}. \tag{22}$$

Owing to its periodicity the transmission coefficient has been represented by a Fourier series. The grating is illuminated by a plane wave $u(\vec{x}) = u_0 e^{i(\vec{k}\vec{x}-\omega t)}$. In our special case of a wave propagating in z-direction perpendicular to the plane of the grating only the z-component of the

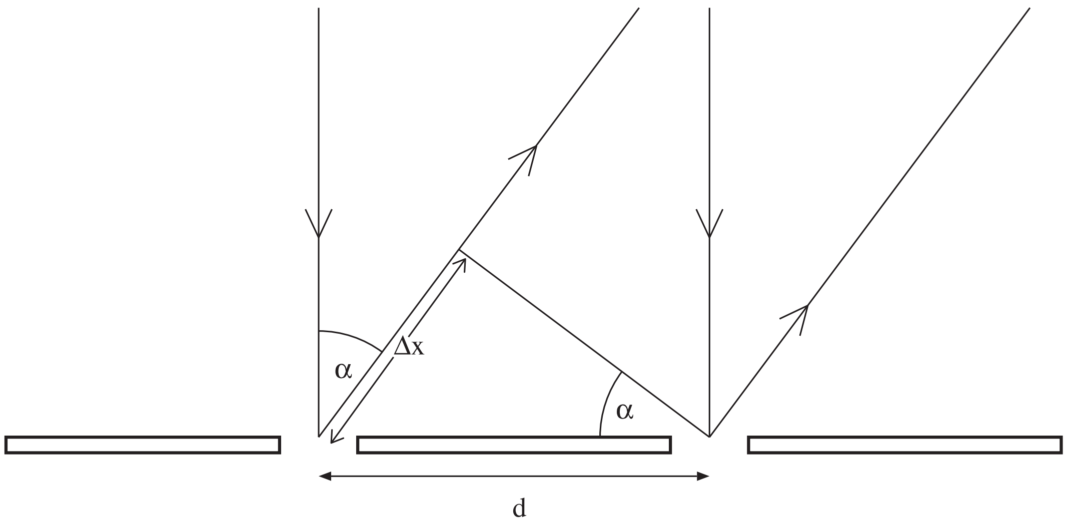

Figure 1: Diffraction on a grating

vector $\vec{k}$ is unequal to zero. Behind the grating ($z > 0$) one gets

$$u(x, y, z) = t(x)\, u_0 e^{i(k_z z - \omega t)} \tag{23}$$

or

$$u(x, y, z) = \sum_{n=-\infty}^{\infty} \hat{t}_n e^{2\pi i n x/d} u_0 e^{i(k_z z - \omega t)} = u_0 \sum_{n=-\infty}^{\infty} \hat{t}_n e^{i(2\pi n x/d + k_z z - \omega t)} \tag{24}$$

for the wave. Equation (24) describes a wave-field, which consists of plane waves with horizontal space-frequencies $k_x = 2\pi n/d$. While the incoming field is a plane wave with wave vector $\vec{k} = (0, 0, k_z)$, the diffracted field consists of a superposition of discrete plane waves which form an angle α with $\vec{k}$. The angle α satisfies the relation $\sin \alpha_n = k_{x,n}/k$. From the form of $k_{x,n}$ one can derive the Bragg-condition

$$\sin(\alpha) = \frac{n\lambda}{d}, \tag{25}$$

which is consistent with the previous approach. Additionally one can calculate the distribution of the relative light intensities ($I_n = |\hat{t}_n|^2$, where $\sum_{n=-\infty}^{\infty} I_n = 1$) which are diffracted into the different spatial angles. From this derivation one sees that the periodicity of the different orders of diffraction is simply defined by the periodicity of the grating. The precise configuration of the grating (like in the examples of the transmission grating or the phase grating) defines the intensity distribution of the different orders. The Fourier method can be extended to a general transmission function without further complications.

1.6 Imaging properties of lenses

In order to understand the imaging properties of lenses, one can take into account three different light beams that start from an object (point) (see Fig. 2).

1. Parallel-beam: Light beams that run parallel to the optical axis in front of the lens, will go through the focal point behind the lens.

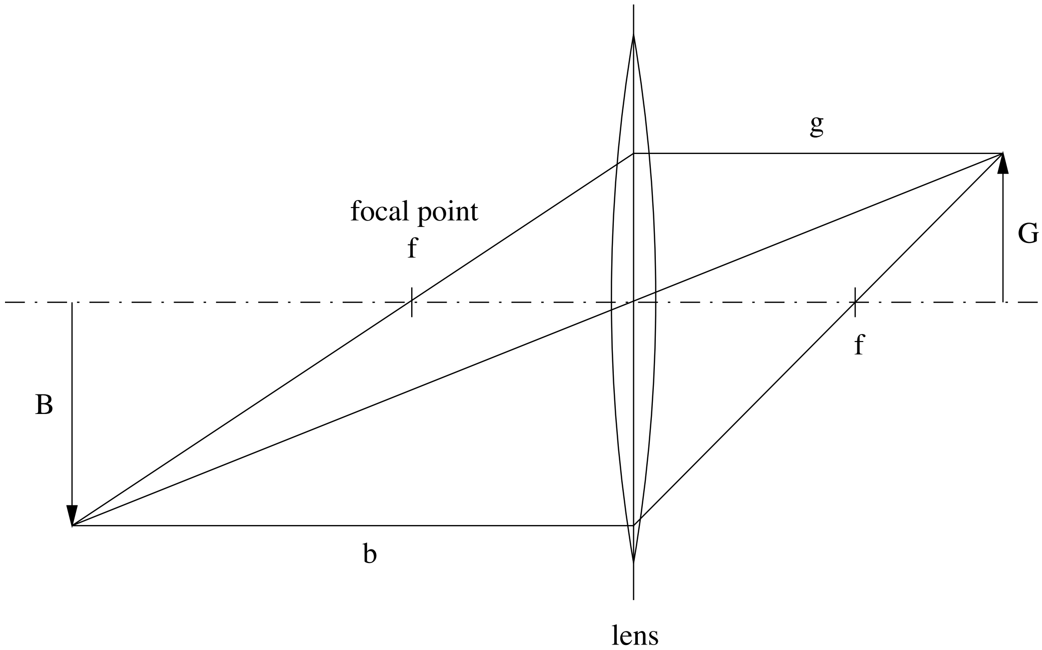

Figure 2: Construction of a picture projected by a collecting lens

2. Center-beam: Light beams that run through the center of the lens are not deflected.
3. Focal-point-beam: light beams, that run through the focal point on the object side of a lens will be directed parallel to the optical axis.

From the second item and the intercept theorem we can see that

$$\frac{B}{G} = \frac{b}{g}, \tag{26}$$

and in analogy with the third item and the intercept theorem we find

$$\frac{B}{G} = \frac{f}{f-g}. \tag{27}$$

By combining equations (26) and (27) one gets the lens equation

$$\frac{1}{f} = \frac{1}{g} + \frac{1}{b}. \tag{28}$$

Using this equation one can describe the projection behavior of a lens completely. One should take into account, that for the case $g < f$ no projection is possible. Instead of getting a real picture (projection), one gets a virtual picture on the side of the object. In the case $g = f$, b gets infinitely big, so that one point in the focal plane on one side of the lens gets transformed into a collimated beam on the other side of the lens. The angle α, which is measured between this beam and the optical axis, is proportional to the transverse distance G of the point to the focal point according to $G \approx \alpha f$.

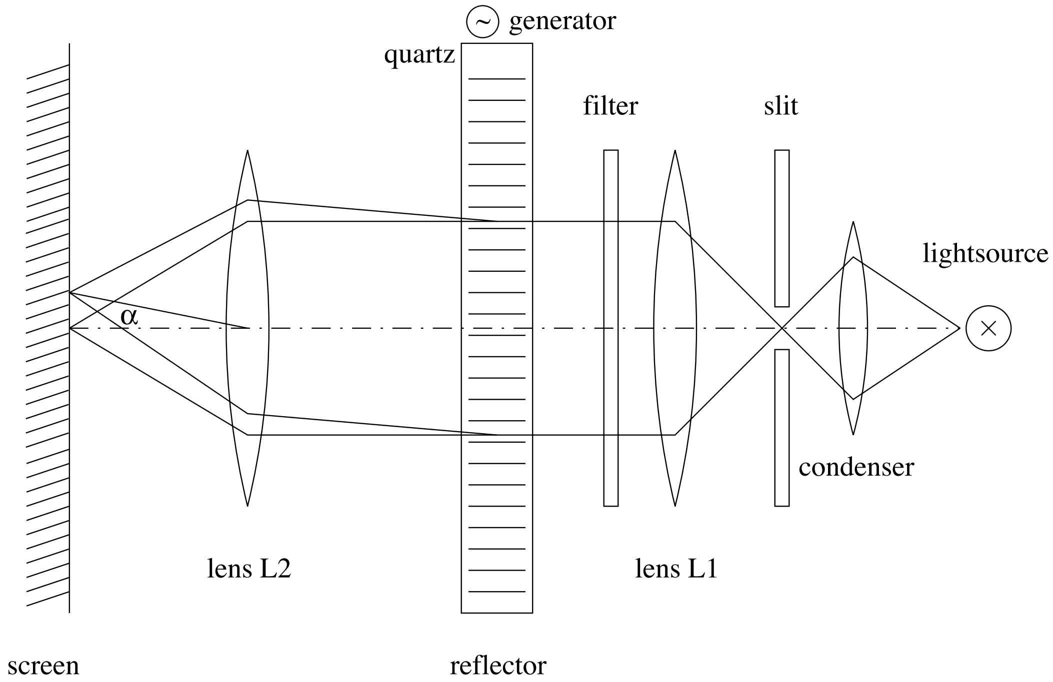

Figure 3: Diffraction method with two lenses

2 The experiment

2.1 General remarks

The propagation velocity c of an ultrasound wave in water will be calculated by measuring the wavelength λ and the frequency ν of the sound waves propagating in it. The sound velocity follows from the relation $c = \lambda\nu$. The frequency of the sound will be measured by a frequency counter. The wavelength will be determined optically by using two different methods: the diffraction- and the schlieren-method.

2.1.1 Diffraction method with two lenses

With this method the diffraction of a plane wave can be measured by projecting the diffracted wave on to a screen using a lens. Figure 3 shows the experimental setup.

The light that emerges from the slit will be parallelized by the lens L1 (the beam can be approximated by a plane wave with sharply defined k-vector parallel to the optical axis) and passes the cuvette (first without ultrasound). The lens L2 focuses an out-going plane wave with a particular k-vector to a point on the screen (see Fig. 3). Each particular angle α between $\vec{k}$ and the optical axis corresponds to a unique point on the screen. If you switch on the ultrasound, the standing wave pattern emerges in the cuvette, if the length of the cuvette is an integer multiple of half the wavelength. The light that passes through the cuvette will be diffracted at the grating. The lens L2 focuses the diffracted light onto the screen. Smaller wavelengths of the standing wave in the cuvette lead to larger spacing of interference maxima on the screen. In this sense, the pattern on the screen is the Fourier transform of the standing wave in the cuvette.

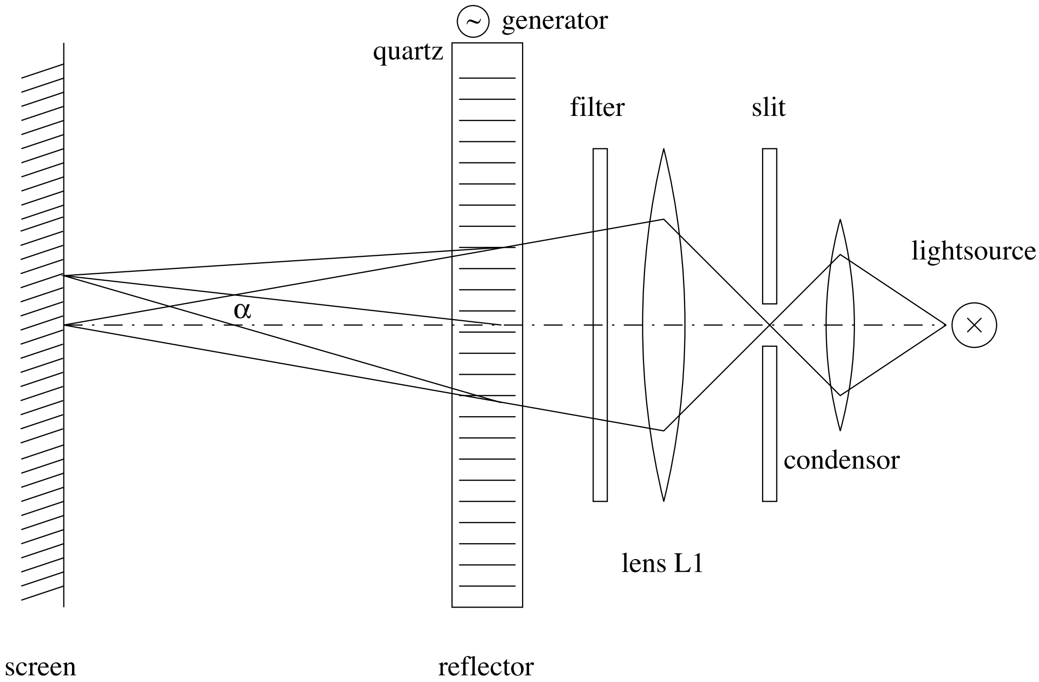

Figure 4: Diffraction method with one lens

Question 1: Which quantities are needed to calculate the wavelength of the sound from the picture on the screen? Explain your reasoning.

2.1.2 Diffraction method with one lens

The diffraction method also works without lens L2 (see Fig. 4). In this case the light has to be focused on the screen by lens L1. Apart from that, the diffraction pattern on the screen appears in the same way as described by the method with two lenses. The main advantage of this method consists in a bigger achievable magnification of the diffraction image.

Question 2: Which quantities are needed to calculate the wavelength of the sound from the picture on the screen in this case? Explain your reasoning.

2.1.3 Schlieren method with two lenses

In contrast to the diffraction method, the schlieren method projects the standing wave directly onto the screen. The Fourier-transformation of the grating does not take place. The setup (see Fig. 5) is similar to the diffraction method in Fig. 3 with the difference that the lens L2 does not focus the parallel light, that serves just to illuminate the standing wave, onto the screen, but onto the blind. The blind blocks the light that is not scattered by the standing wave, because otherwise this light would interfere with the wave image and would worsen the contrast. The diffracted light passes the blind undisturbed. If the distances between optical elements are chosen correctly, the scattered light that is emitted by every wave lobe is projected onto the screen sharply. In this way it is possible to obtain a direct picture of the standing wave. The advantage of this method compared to the diffraction method is the possible magnification of the

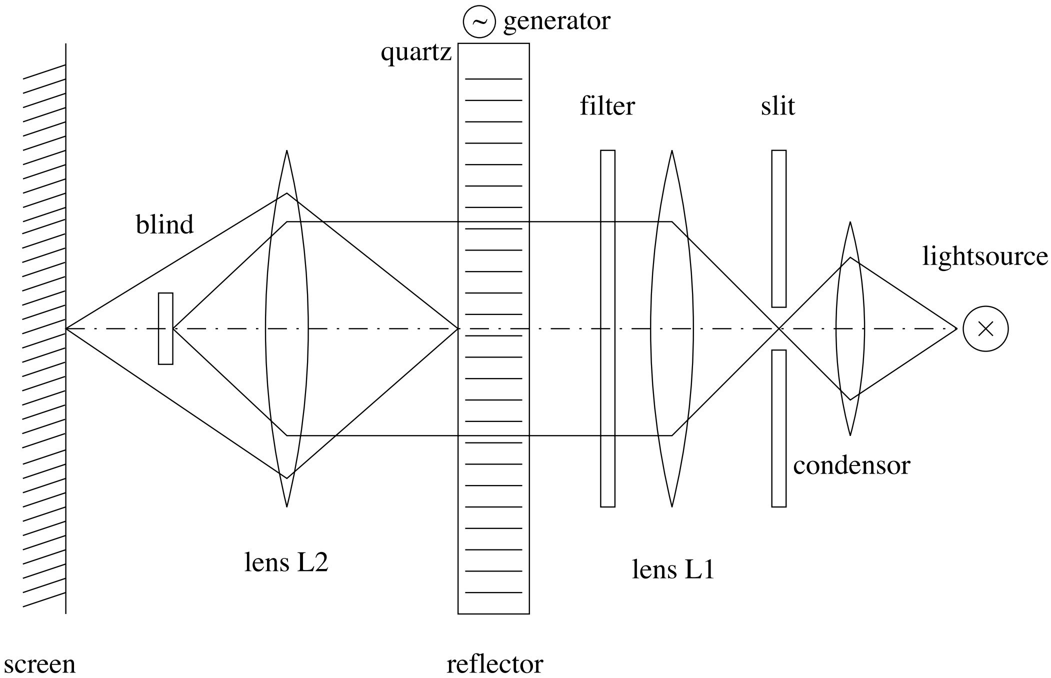

Figure 5: Schlieren method with two lenses

interference pattern. The magnification factor and the distances cuvette–lens and lens–screen follow from the lens equations (chapter 1.6). The focal length of the lens L2 and the distances of the screen from the lens L2 can be chosen freely.

Question 3: What are the minimum distances cuvette–lens L2 and lens L2–screen that produce a sharp picture? On which quantities does the magnification factor depend?

Question 4: In this case the sound waves are directly imaged, and the exposure time is typically much longer than the wave period. In the limiting case of long exposure times, how will the recorded image differ from the real standing wave pattern?

2.1.4 Schlieren method with one lens

In analogy to the diffraction method it is also possible to obtain a schlieren picture with only one lens (see Fig. 6). The larger magnification is accompanied by worse image quality.

2.2 Performing the measurement

A green LED light source is used for all the experiments. For the interference experiments we need the light to be monochromatic and the light source to be point-shaped. The former is achieved by isolating part of the green spectrum by using an interference filter. The latter is obtained by focusing the the light on a small slit by using a lens (condensor, use the smallest focal length available).

Question 5: Why does the light source have to be monochromatic (for the diffraction experiment) and point-shaped (for both methods)?

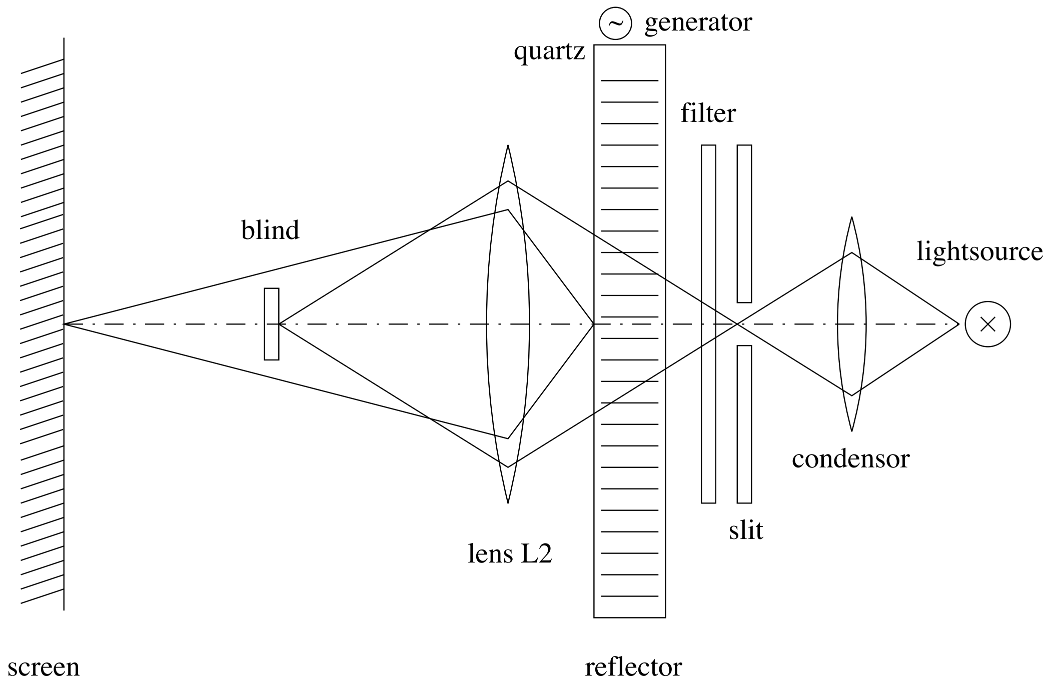

Figure 6: Schlieren method with one lens

3 Photography

You will directly project the obtained interference patterns onto the chip of a digital camera. For this purpose you will use the camera without objective, as shown in Fig. 7, and mount it directly on the optical bench. You will need to find the pixel size of the camera you are using online.

In the following, we will discuss some important features of this camera in order to obtain high-quality pictures. First, you set it to M to have manual access to all the available functions. This mode is set with the small wheel next to the ON/OFF-switch (see Fig. 8).

3.1 Aperture and shutter speed

In manual mode, you can change aperture and shutter speed using the "Q"-button (see Fig. 9) and the cursor. It is advantageous to set the aperture to the smallest possible value, whereas the shutter speed has to match the surrounding ambient light.

3.2 Live-mode

The small button with the red dot next to it, directly next to the display, allows you to go into live-mode (see Fig. 9). While you normally have to look through the small window on top of the camera, you can now look at the image on the display in this mode which significantly simplifies your adjustments.

Figure 7: Canon PowerShot 1100D

Figure 8: Setting the manual mode

3.3 Zoom

Beware that without objective you do not have access to an optical zoom, meaning that you can only magnify the displayed image for preview. You have to do any zooming operation on the computer.

3.4 Sensitivity

Sensitivity (ISO) can also be changed in the menu accessed by the "Q"-button by entering the ISO-menu. It is recommended to set the sensitivity to a value as low as possible to reduce noise.

Figure 9: Setting the parameters and the live-mode